

ON  $n$ -STRONGLY GORENSTEIN RINGS

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ABSTRACT. This paper introduces and studies a particular subclasses of the class of commutative rings with finite Gorenstein global (resp., weak) dimensions.

## 1. INTRODUCTION

Throughout this paper, all rings are commutative with identity, and all modules are unitary.

Let  $R$  be a ring, and let  $M$  be an  $R$ -module. As usual we use  $\text{pd}_R(M)$ ,  $\text{id}_R(M)$  and  $\text{fd}_R(M)$  to denote, respectively, the classical projective dimension, injective dimension and flat dimension of  $M$ . We use  $\text{gldim}(R)$  and  $\text{wdim}(R)$  to denote, respectively, the classical global and weak global dimension of  $R$ .

For two-sided Noetherian ring  $R$ , Auslander and Bridger [1] introduced the  $G$ -dimension,  $\text{Gdim}_R(M)$ , for every finitely generated  $R$ -module  $M$ . They showed that there is an inequality  $\text{Gdim}_R(M) \leq \text{pd}_R(M)$  for all finite  $R$ -modules  $M$ , and the equality holds if  $\text{pd}_R(M)$  is finite.

Several decades later, Enochs and Jenda [8, 9] defined the notion of Gorenstein projective dimension ( $G$ -projective dimension for short), as an extension of  $G$ -dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension ( $G$ -injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [11] introduced the Gorenstein flat dimension. Some references are [3, 6, 7, 8, 9, 11, 16].

Recall that an  $R$ -module  $M$  is called Gorenstein projective if, there exists an exact sequence of projective  $R$ -modules:

$$\mathbf{P} : \dots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} \dots$$

such that  $M \cong \text{Im}(f_0)$  and such that the operator  $\text{Hom}_R(-, Q)$  leaves  $\mathbf{P}$  exact whenever  $Q$  is projective. The resolution  $\mathbf{P}$  is called a complete projective resolution. In particular, if for each  $i$  we have  $P_i = P^i = P$  and  $f^i = f_i = f$ , the  $R$  module  $M$  is called strongly Gorenstein projective (please see [2]).

The (strongly) Gorenstein injective  $R$ -modules are defined dually.

And an  $R$ -module  $M$  is called Gorenstein flat if, there exists an exact sequence of

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flat left  $R$ -modules:

$$\mathbf{F} : \dots \rightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} F^0 \xrightarrow{f^0} F^1 \xrightarrow{f^1} \dots$$

such that  $M \cong \text{Im}(f_0)$  and such that the operator  $- \otimes_R I$  leaves  $F$  exact whenever  $I$  is an injective  $R$ -module. The resolution  $\mathbf{F}$  is called complete flat resolution. In particular, if for each  $i$  we have  $F_i = F^i = F$  and  $f^i = f_i = f$ , the  $R$  module  $M$  is called strongly Gorenstein flat (please see [2]).

The Gorenstein projective, injective and flat dimensions are defined in term of resolution and denoted by  $Gpd(-)$ ,  $Gid(-)$  and  $Gfd(-)$  respectively (see [6, 12, 16]).

In [3], the authors prove the equality:

$$\sup\{Gpd_R(M) | M \text{ is an } R\text{-module}\} = \sup\{Gid_R(M) | M \text{ is an } R\text{-module}\}$$

They called the common value of the above quantities the Gorenstein global dimension of  $R$  and denoted it by  $Ggldim(R)$ . Similarly, they set

$$wGgldim(R) = \{Gfd_R(M) | M \text{ is an } R\text{-module}\}$$

which they called the weak Gorenstein global dimension of  $R$ . Note that the results and the notation in [3] are in the non-commutative case. In this paper our rings are commutative and so the left and right (Gorenstein) homological dimensions can be identified.

Recently, in [19], particular modules of finite Gorenstein projective, injective and flat dimensions are defined as follows:

**Definitions 1.1.** Let  $n$  be a positive integer.

- (1) An  $R$ -module  $M$  is said to be strongly  $n$ -Gorenstein projective, if there exists a short exact sequence  $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$  where  $pd_R(P) \leq n$  and  $Ext_R^{n+1}(M, Q) = 0$  whenever  $Q$  is projective.
- (2) An  $R$ -module  $M$  is said to be strongly  $n$ -Gorenstein injective, if there exists a short exact sequence  $0 \rightarrow M \rightarrow I \rightarrow M \rightarrow 0$  where  $id_R(I) \leq n$  and  $Ext_R^{n+1}(E, M) = 0$  whenever  $E$  is injective.
- (3) An  $R$ -module  $M$  is said to be strongly  $n$ -Gorenstein flat, if there exists a short exact sequence  $0 \rightarrow M \rightarrow F \rightarrow M \rightarrow 0$  where  $fd_R(F) \leq n$  and  $Tor_R^{n+1}(M, I) = 0$  whenever  $I$  is injective.

Clearly, the strongly 0-Gorenstein projective, injective and flat modules are the strongly Gorenstein projective, injective and flat modules respectively ([3, Propositions 2.9 and 3.6]).

In this paper, we investigate these modules to characterize new classes of rings with finite (resp., weak) Gorenstein global dimensions. In [19], the authors prove the following Proposition:

**Proposition 1.2.** [19, Proposition 2.16] *Let  $R$  be a ring. The following statements are equivalent:*

- (1) *Every module is strongly  $n$ -Gorenstein projective.*
- (2) *Every module is strongly  $n$ -Gorenstein injective.*

Hence, we give the following definitions:

**Definitions 1.3.** Let  $n$  be a positive integer.

- (1) A ring  $R$  is called  $n$ -strongly Gorenstein ( $n$ -SG ring for a short) if  $R$  satisfies one of the equivalent conditions of Proposition 1.2.
- (2) A ring  $R$  is called weakly  $n$ -strongly Gorenstein ( $n$ -wSG for a short) if every  $R$ -module is strongly  $n$ -Gorenstein flat.

The rings 0-SG and 1-SG are already studied in [5, 18] over which they are called strongly Gorenstein semi-simple and hereditary rings respectively. Clearly, by definition, every  $n$ -SG (resp.,  $n$ -wSG) ring is  $m$ -SG (resp.,  $m$ -wSG) whenever  $n \leq m$ .

After given some characterizations of the  $n$ -SG and  $n$ -wSG rings (see Propositions 2.1 and 2.2), we will see that for any ring  $R$  we have:

$$\begin{aligned} \text{gldim}(R) \leq n &\implies R \text{ is } n\text{-SG} \implies \text{Ggldim}(R) \leq n, \text{ and} \\ \text{wldim}(R) \leq n &\implies R \text{ is } n\text{-wSG} \implies \text{wGgldim}(R) \leq n. \end{aligned}$$

We will also give examples which show that the inverse implications are not true in the general case (see Examples 2.11 and 2.12). After that, Theorem 2.6 proves that:

$$R \text{ is an } n\text{-SG ring} \implies R \text{ is an } n\text{-wSG ring}$$

with equivalence if  $R$  is Noetherian or perfect with finite Gorenstein global dimension. But in the general case the inverse implication is not true as shown by Example 2.12.

## 2. MAIN RESULTS

Note that the structure of the  $n$ -SG rings (resp.,  $n$ -wSG rings) depends of two variants. The first one is the (resp., weak) Gorenstein global dimension of these rings and the second is the form of their modules.

**Proposition 2.1.** *For a ring  $R$  and a positive integer  $n$ , the following statements are equivalent:*

- (1)  $R$  is  $n$ -SG ring.
- (2)  $\text{Ggldim}(R) \leq n$  and for every  $R$ -module  $M$  there exists a short exact sequence  $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$  where  $\text{pd}_R(P) < \infty$ .
- (3)  $\text{Ggldim}(R) < \infty$  and for every  $R$ -module  $M$  there exists a short exact sequence  $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$  where  $\text{pd}_R(P) \leq n$ .

*Proof.*  $1 \Rightarrow 2$ . Clear since for every  $n$ -SG ring  $R$  we have  $\text{Ggldim}(R) \leq n$  (by [19, Proposition 2.2(1)]).

$2 \Rightarrow 3$ . Follows directly from [3, Corollary 2.7].

$3 \Rightarrow 1$ . Follows from [19, Proposition 2.10]. □

Similarly for the  $n$ -wSG rings we have the following result:

**Proposition 2.2.** *Let  $R$  be a ring,  $n$  a positive integer, and consider the following assertions:*

- (1)  $R$  is  $n$ -wSG ring.
- (2)  $\text{wGgldim}(R) \leq n$  and for every  $R$ -module  $M$  there exists a short exact sequence  $0 \rightarrow M \rightarrow F \rightarrow M \rightarrow 0$  where  $\text{fd}_R(F) < \infty$ .
- (3)  $\text{wGgldim}(R) < \infty$  and for every  $R$ -module  $M$  there exists a short exact sequence  $0 \rightarrow M \rightarrow F \rightarrow M \rightarrow 0$  where  $\text{fd}_R(F) \leq n$ .
- (4) For every  $R$ -module  $M$  there exists a short exact sequence  $0 \rightarrow M \rightarrow F \rightarrow M \rightarrow 0$  where  $\text{fd}_R(F) \leq n$ .

Then,  $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$  with equivalence if  $R$  is coherent.

To prove this Proposition we need the following Lemmas:

**Lemma 2.3.** *Let  $R$  be a coherent ring. The following assertions are equivalent:*

- (1)  $wGgldim(R) \leq n$
- (2)  $fd_R(I) \leq n$  for every injective  $R$ -module  $I$ .

*Proof.* Follows by combining the equivalence [13, Theorem 7(1  $\Leftrightarrow$  2)] and the equality [14, Theorem 3.7(1=2)].  $\square$

**Lemma 2.4.** *For every  $R$ -module  $M$  we have  $Gfd_R(M) \leq fd_R(M)$  with equality if  $fd_R(M) < \infty$ .*

*Proof.* In first note that  $Gfd_R(M) \leq m$  implies that  $Tor_R^i(M, I) = 0$  for every injective  $R$ -module  $I$  and each  $i > m$ . Indeed, consider an  $m$ -step flat resolution of  $M$  as follows:

$$0 \rightarrow G \rightarrow F_m \rightarrow \dots \rightarrow F_1 \rightarrow M \rightarrow 0$$

Clearly  $G$  is Gorenstein flat. Then, by [16, Theorem 3.6],  $Tor_R^i(G, I) = 0$  for every injective  $R$ -module  $I$  and each  $i > 0$ . Therefore,  $Tor_R^{i+m}(M, I) = 0$ , as desired.

The first inequality of this Lemma holds from the fact that every flat module is Gorenstein flat. Now, suppose that  $fd_R(M) = n > 0$  and suppose by absurd that  $Gfd_R(M) < n$ . Thus, there exists an  $R$ -module  $N$  such that  $Tor_R^n(M, N) \neq 0$ . Note that  $N$  can not be injective (since  $Gfd_R(M) < n$ ). Hence, pick a short exact sequence  $0 \rightarrow N \rightarrow I \rightarrow I/N \rightarrow 0$  where  $I$  is injective. Then,  $0 \neq Tor_R^n(M, N) = Tor_R^{n+1}(M, I/N)$ . Absurd since  $fd_R(M) = n$ . This contradiction finish the proof.  $\square$

*Proof of Proposition 2.2.*  $1 \Rightarrow 2$ . Follows from the fact that every strongly  $n$ -Gorenstein flat module has a Gorenstein flat dimension  $\leq n$  (by [19, Proposition 3.2(1)]).

$2 \Rightarrow 3$ . Let  $F$  be an  $R$ -module such that  $fd_R(M) < \infty$ . We have  $Gfd_R(F) \leq n$  since  $wGgldim(R) \leq n$ . Then, by Lemma 2.4,  $fd_R(F) \leq n$ . Hence, the following implication is immediate.

$3 \Rightarrow 1$  Let  $M$  be an arbitrary  $R$ -module. For such module there is an exact sequence  $(\star) \ 0 \rightarrow M \rightarrow F \rightarrow M \rightarrow 0$  where  $fd_R(F) \leq n$ . On the other hand,  $m := Gfd_R(M) < \infty$  since  $wGgldim(R) < \infty$ . Thus, from the note in the proof of Lemma 2.4, for every injective  $R$ -module  $I$  we have  $Tor_R^i(M, I) = 0$  for all  $i > m$ . So, from  $(\star)$ , we have  $Tor_R^{n+1}(M, I) = Tor_R^{n+2}(M, I) = \dots = Tor_R^{n+m+1}(M, I) = 0$  since  $n + m + 1 > m$ . Thus,  $M$  is a strongly  $n$ -Gorenstein flat module, as desired.

$3 \Rightarrow 4$ . Obvious.

Now assume that  $R$  is coherent.

$4 \Rightarrow 1$ . Let  $I$  be an injective  $R$ -module. By hypothesis, there is an exact sequence  $0 \rightarrow I \rightarrow F \rightarrow I \rightarrow 0$  where  $fd_R(F) \leq n$ . Clearly this exact sequence splits. Thus,  $I \oplus I \cong F$ . Hence,  $fd_R(I) \leq n$ . Consequently, from Lemma 2.3 and since  $R$  is coherent,  $wGgldim(R) \leq n$ . Thus, for each  $R$ -module  $M$  and every injective  $R$ -module  $I$  we have  $Tor_R^{n+1}(M, I) = 0$  (since  $Gfd_R(M) \leq n$ ). Thus, adding this fact to the hypothesis condition, we conclude that every  $R$ -module is strongly  $n$ -Gorenstein flat as desired.  $\square$

**Remark 2.5.** Let  $R$  be an  $n$ -SG ring (resp.,  $n$ -wSG ring). We have:

- (1) From Propositions 2.1 and 2.2,  $Ggldim(R) \leq n$  (resp.,  $wGgldim(R) \leq n$ ).
- (2) Using [3, Corollary 1.2],  $gldim(R) \leq n$  (resp.,  $wdim(R) \leq n$ ) if, and only if,  $wdim(R) < \infty$ .

Recall that a ring  $R$  is called perfect if, every flat  $R$ -module is projective.

**Theorem 2.6.** *Every  $n$ -SG ring is  $n$ -wSG with equivalence in the following two cases:*

- (1)  $R$  is Noetherian.
- (2)  $R$  is perfect with finite Gorenstein global dimension.

To prove this theorem we need the following Lemma.

**Lemma 2.7.** *For any arbitrary ring  $R$  we have:  $wGgldim(R) \leq Ggldim(R)$  with equality in the following two cases:*

- (1)  $R$  is Noetherian.
- (2)  $R$  is perfect with finite Gorenstein global dimension.

*Proof.* To prove the desired inequality, the standard argument shows that it suffices to prove that every Gorenstein projective  $R$ -module is Gorenstein flat provided  $Ggldim(R) < \infty$ . So, let  $\mathbf{P}$  be a complete projective resolution. We have to prove that  $I \otimes_R \mathbf{P}$  is exact for every injective  $R$ -module  $I$ . Since  $Ggldim(R) < \infty$  and by [3, Corollary 2.7] we have  $fd_R(I) < \infty$ . Let  $I^* := Hom_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$  be the character of  $I$ . By [20, Theorem 3.52],  $id_R(I^*) = fd_R(I) < \infty$ . Again, by [3, Corollary 2.7],  $pd_R(I^*)$  is finite. Consequently, by [16, Proposition 2.3],  $Hom_R(\mathbf{P}, I^*)$  is exact. By adjointness,  $Hom_{\mathbb{Z}}(I \otimes_R \mathbf{P}, \mathbb{Q}/\mathbb{Z}) = Hom_R(\mathbf{P}, I^*)$ . Then,  $I \otimes_R \mathbf{P}$  is exact, as desired.

If  $R$  is Noetherian, the converse inequality follows from the equivalence [12, Theorem 12.3.1(3  $\Leftrightarrow$  4)].

Now suppose that  $R$  is perfect with  $n := Ggldim(R) < \infty$ . We prove that every Gorenstein flat  $R$ -module is Gorenstein projective. Let  $M$  be an arbitrary Gorenstein flat module. By definition we can pick an  $n$ -step right flat resolution as follows:

$$0 \rightarrow M \rightarrow F_1 \rightarrow F_1 \rightarrow \dots \rightarrow F_n \rightarrow G$$

where all  $F_i$  are flat and so projective since  $R$  is perfect. But  $Gpd_R(G) \leq n$ . Thus, using the equivalence [16, Theorem 2.20( $i \Leftrightarrow iv$ )], we conclude that  $M$  is Gorenstein projective as desired. Consequently,  $wGgldim(R) \leq Ggldim(R)$  and this finish the proof.  $\square$

*Proof of Theorem 2.6.* Let  $R$  be an  $n$ -SG ring. Clearly  $Ggldim(R) \leq n$ . Then, by Lemma 2.7,  $wGgldim(R) \leq n$ . Now, let  $M$  be an arbitrary  $R$ -module. By hypothesis,  $M$  is strongly  $n$ -Gorenstein projective. Then, there is a short exact sequence  $0 \rightarrow M \rightarrow P \rightarrow M \rightarrow 0$  where  $pd_R(M) \leq n$ . So, by Proposition 2.2,  $R$  is an  $n$ -wSG ring, as desired.

Now, let  $R$  be an  $n$ -wSG ring. Then,  $wGgldim(R) \leq n$ . So, if  $R$  is Noetherian or perfect with finite Gorenstein global dimension then  $Ggldim(R) \leq n$  (by Lemma 2.7). Now, let  $M$  be an arbitrary  $R$ -module. By hypothesis, there is an exact sequence  $\rightarrow M \rightarrow F \rightarrow M \rightarrow 0$  where  $fd_R(F) \leq n$ . Using [3, Corollary 2.7], we have  $pd_R(F) \leq n$ . So, by Proposition 2.1,  $R$  is an  $n$ -SG ring, as desired.  $\square$

**Theorem 2.8.** *Let  $\{R_i\}_{i=1}^m$  be a family of rings and set  $R := \prod_{i=1}^m R_i$ . Then,  $R$  is an  $n$ -SG ring if, and only if,  $R_i$  is an  $n$ -SG ring for each  $i \in I$ . Moreover, if  $R_i$  is Coherent for each  $i$ , then  $R$  is an  $n$ -wSG ring if, and only if,  $R_i$  is an  $n$ -wSG ring for each  $i \in I$ .*

*Proof.* By induction on  $m$  it suffices to prove the assertion for  $m = 2$ . First suppose that  $R_1 \times R_2$  is  $n$ -SG ring. We claim that  $R_1$  is an  $n$ -SG ring. Let  $M$  be an arbitrary  $R_1$  module.  $M \times 0$  can be viewed as an  $R_1 \times R_2$ -module. For such module and since  $R_1 \times R_2$  is an  $n$ -SG ring, there is an exact sequence  $0 \rightarrow M \times 0 \rightarrow P \rightarrow M \times 0 \rightarrow 0$  where  $pd_{R_1 \times R_2}(P) \leq n$ . Thus, since  $R_1$  is a projective  $R_1 \times R_2$  module, by applying  $-\otimes_{R_1 \times R_2} R_1$  to the sequence above, we find the exact sequence of  $R$ -modules:  $0 \rightarrow M \times 0 \otimes_{R_1 \times R_2} R_1 \rightarrow P \otimes_{R_1 \times R_2} R_1 \rightarrow M \times 0 \otimes_{R_1 \times R_2} R_1 \rightarrow 0$ . Clearly  $pd_{R_1}(P \otimes_{R_1 \times R_2} R_1) \leq pd_{R_1 \times R_2}(P) \leq n$ . Moreover, we have the isomorphism of  $R$ -modules:  $M \times 0 \otimes_{R_1 \times R_2} R_1 \cong M \times 0 \otimes_{R_1 \times R_2} (R_1 \times R_2)/(0 \times R_2) \cong M$ . Thus, we obtain an exact sequence of  $R$ -module with the form:  $0 \rightarrow M \rightarrow P \otimes_{R_1 \times R_2} R_1 \rightarrow M \rightarrow 0$ . On the other hand, by [4, Theorem 3.1], we have  $Ggldim(R_1) \leq Ggldim(R_1 \times R_2) \leq n$ . Thus, using Proposition 2.1,  $R_1$  is an  $n$ -SG ring, as desired. By the same argument,  $R_2$  is also an  $n$ -SG ring.

Now, suppose that  $R_1$  and  $R_2$  are  $n$ -SG rings and we claim that  $R_1 \times R_2$  is an  $n$ -SG ring. Let  $M$  be an  $R_1 \times R_2$ -module. We have

$$M \cong M \otimes_{R_1 \times R_2} (R_1 \times R_2) \cong M \otimes_{R_1 \times R_2} ((R_1 \times 0) \oplus (0 \times R_2)) \cong M_1 \times M_2$$

where  $M_i = M \otimes_{R_1 \times R_2} R_i$  for  $i = 1, 2$ . For each  $i = 1, 2$ , there is an exact sequence  $0 \rightarrow M_i \rightarrow P_i \rightarrow M_i \rightarrow 0$  where  $pd_{R_i}(P_i) \leq n$  since  $R_i$  is an  $n$ -SG ring. Thus, we have the exact sequence of  $R_1 \times R_2$ -modules:  $0 \rightarrow M_1 \times M_2 \rightarrow P_1 \times P_2 \rightarrow M_1 \times M_2 \rightarrow 0$ . On the other hand,  $pd_{R_1 \times R_2}(P_1 \times P_2) = \sup\{pd_{R_i}(P_i)\}_{1,2} \leq n$  (by [17, Lemma 2.5(2)]). Moreover, from [4, Theorem 3.1],  $Ggldim(R_1 \times R_2) = \sup\{Ggldim(R_i)\}_{1,2} \leq n$ . Thus, from Proposition 2.1,  $R_1 \times R_2$  is an  $n$ -SG ring, as desired.

If  $R_i$  is coherent for each  $i = 1, 2$ , by [15, Theorem 2.4.3],  $R_1 \times R_2$  is coherent. So, using [4, Theorem 3.5 and Lemma 3.7], by the same reasoning that above, we prove the result for the  $n$ -wSG rings.  $\square$

Let  $T := R[X_1, X_2, \dots, X_n]$  the polynomial ring in  $n$  indeterminates over  $R$ . If we suppose that  $T$  is an  $m$ -SG ring, it is easy to see by [4, Theorem 2.1], that  $n \leq m$ .

**Theorem 2.9.** *If  $R[X_1, X_2, \dots, X_n]$  is an  $m$ -SG ring, then  $R$  is an  $(m - n)$ -SG ring.*

*Proof.* By induction on  $n$  it suffices to prove the result for  $n = 1$ . So, suppose that  $R[X]$  is an  $m$ -SG ring. Let  $M$  be an arbitrary  $R$ -module. For the  $R[X]$ -module  $M[X] := M \otimes_R R[X]$  there is an exact sequence of  $R[X]$ -modules  $0 \rightarrow M[X] \rightarrow P \rightarrow M[X] \rightarrow 0$  where  $pd_{R[X]}(P) \leq m$ . Applying  $-\otimes_{R[X]} R$  to the short exact sequence above and seeing that  $M \cong_R M[X] \otimes_{R[X]} R$ , we obtain a short exact sequence of  $R$ -modules with the form  $0 \rightarrow M \rightarrow P \otimes_{R[X]} R \rightarrow M \rightarrow 0$  (see that  $R$  is a projective  $R[X]$ -module). Moreover,  $pd_R(P \otimes_{R[X]} R) \leq pd_{R[X]}(P) < \infty$ . On the other hand,  $Ggldim(R) = Ggldim(R[X]) - 1 \leq m - 1$  (by [4, Theorem 2.1]). Hence, by Proposition 2.1,  $R$  is an  $(m - 1)$ -SG ring, as desired.  $\square$

**Theorem 2.10.** *If  $R[X_1, X_2, \dots, X_n]$  is a coherent  $m$ -wSG ring, then  $R$  is an  $(m - n)$ -wSG-ring.*

*Proof.* Note in first that for every  $i \leq n$ , the polynomial ring  $R[X_1, \dots, X_i]$  is coherent (by using [15, Theorem 4.1.1(1)]). By the induction on  $n$ , it suffices to prove the result for  $n = 1$ . Using [4, Theorem 2.11] and Proposition 2.2, the proof is similar to the proof of Theorem 2.9.  $\square$

Trivial examples of the  $n$ -SG-ring (resp.,  $n$ -wSG ring) are the rings with global dimension (resp., weak global dimension)  $\leq n$ . The following example give a new family of commutative  $n$ -SG rings (resp.,  $n$ -wSG rings) with infinite weak global dimension.

*Example 2.11.* Consider the non semi-simple quasi-Frobenius rings  $R_1 := K[X]/(X^2)$  and  $R_2 := K[X]/(X^3)$  where  $K$  is a field and let  $S$  be a non Noetherian ring such that  $\text{gldim}(S) = n$ . Then,

- (1)  $\text{Ggldim}(R_1) = \text{Ggldim}(R_2) = 0$  and  $R_1$  is 0-SG ring but  $R_2$  is not.
- (2)  $R_1 \times S$  is a non Noetherian  $n$ -SG (and so  $n$ -wSG) ring with infinite weak global dimension.
- (3)  $\text{Ggldim}(R_2 \times S) = n$  but  $R_2 \times S$  is not an  $n$ -SG ring (a non Noetherian ring) with infinite weak global dimension.
- (4)  $\text{wGgldim}(R_2[X_1, \dots, X_n]) = \text{Ggldim}(R_2[X_1, \dots, X_n]) = n$  but  $R_2[X_1, \dots, X_n]$  is neither  $n$ -SG ring nor  $n$ -wSG ring with infinite weak global dimension.

*Proof.* From [5, Corollary 3.9] and [3, Proposition 2.6],  $\text{Ggldim}(R_1) = \text{Ggldim}(R_2) = 0$  and  $R_1$  is 0-SG ring but  $R_2$  is not. So, (1) is clear. Moreover  $R_1$  and  $R_2$  have infinite weak global dimensions. By [4, Theorems 2.1 and 3.1] and Lemma 2.7 and the fact that  $R_2$  is Noetherian, it is easy to see that,

- $\text{Ggldim}(R_2 \times S) = n$ , and
- $\text{wGgldim}(R_2[X_1, \dots, X_n]) = \text{Ggldim}(R_2[X_1, \dots, X_n]) = n$ .

And using Theorems 2.8 and 2.9, we conclude that  $R_1 \times S$  is an  $n$ -SG (and so  $n$ -wSG by Theorem 2.6) and that  $R_2[X_1, \dots, X_n]$  is neither  $n$ -SG ring nor  $n$ -wSG ring (by Theorem 2.6 since  $R_2$  is Noetherian). Hence, (2), (3) and (4) hold.  $\square$

With rings with finite weak global dimension, it is clear that left implication of Theorem 2.6 is not true. The following Example shows the same thing with rings with infinite weak global dimensions.

*Example 2.12.* Consider the non semi-simple quasi-Frobenius rings  $R_1 := K[X]/(X^2)$  and  $R_2 := K[X]/(X^3)$  where  $K$  is a field and a family of coherent rings  $\{S_n\}_{n \in \mathbb{N}}$  such that  $n = \text{wdim}(S_n) < \text{gldim}(S_n)$  (for example  $S_n := S_0[X_1, X_2, \dots, X_n]$  where  $S_0$  is a non-Noetherian Von Neumann regular ring). For every positive integer  $n$ , set  $R_1^n := R_0 \times S_n$  and  $R_2^n := R_1 \times S_n$ . Then,

- (1)  $R_0^n$  is an  $n$ -wSG ring which is not  $n$ -SG ring.
- (2)  $\text{wGgldim}(R_1^n) = n$  but  $R_1^n$  is not an  $n$ -wSG ring.

*Proof.* (1). Since  $\text{wdim}(S_n) = n$ , the ring  $S_n$  is  $n$ -wSG. On the other hand, from [5, Corollary 3.9],  $R_1$  is a 0-SG ring. Then, it is also an  $n$ -SG ring. Hence, from Theorem 2.6,  $R_1$  is an  $n$ -wSG ring. Thus, by Theorem 2.8,  $R_1 \times S_n$  is an  $n$ -wSG ring. But  $\text{gldim}(S_n) > n$  implies that  $\text{Ggldim}(R_1^n) > n$  (by [4, Theorem 3.1]). So, from Proposition 2.1,  $R_1^n$  is not an  $n$ -SG ring, as desired.

(2). From [4, Theorem 3.5], it is clear that  $wGldim(R_2^n) = n$ . And using Theorem 2.8, if  $R_2^n$  is an  $n$ -wSG ring, we conclude that  $R_2$  is an  $n$ -wSG ring and so  $n$ -SG ring since  $R$  is Noetherian. But  $Gldim(R_2) = 0$ . So, from Proposition 2.1,  $R_2$  is a 0-SG ring. Contradiction with Example 2.11.  $\square$

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